The reduction of $\mathbf{T}_1$ to $\mathbf{T}_{11}$, however, is effected by the residual propagator

$$\mathbf{\Gamma}_1 = |\alpha\rangle(W - E_\alpha + \langle\alpha|\mathbf{V}|\alpha\rangle)^{-1}\langle\alpha|\,.$$

These choices correspond to using the quasiparticle formalism for the first two reductions and using the projection operator formalism for the third reduction. As a result one finds that besides Eq. (62) the physical equivalence of the augmented system to the original system requires $W - E_\alpha - \langle\alpha|\mathbf{T}_{11}|\alpha\rangle = -E_\alpha - \langle a|T_1|a\rangle$. Thus the equivalence obtains only in the limit $E_\alpha \gg W$. But it is seen that if the projection operator formalism is used for all three reductions, then the equivalence can be made exact.

We have shown how an elementary particle state can be used to represent a bound state of the system. There is no reason why we cannot reverse the procedure and use a bound state to replace an elementary-particle state. One merely regards Eq. (43) as a complete definition of H, H_0, and V rather than a partial definition of $\mathbf{H}$, $\mathbf{H}_0$, and $\mathbf{V}$. Since $\mathbf{H}_1 = \mathbf{H}_0 + \mathbf{V}_1$ is now the starting point, $\mathbf{V}$ must be defined in terms of $\mathbf{V}_1$

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_1 - \mathbf{V}_1\mathbf{\Gamma}\mathbf{V}_1 \\ \mathbf{\Gamma} &= -(1 - \mathbf{GQV}_1)^{-1}\mathbf{GQ} \\ &= |\alpha\rangle(E_\alpha + \langle\alpha|\mathbf{V}_1|\alpha\rangle - W)^{-1}\langle\alpha|\,. \end{aligned} \tag{66}$$

Equation (62) now plays the role of the definition of the matrix elements of V with respect to the new bound state $|a\rangle$.

$$\begin{aligned} \langle a|V|a\rangle &= \langle\alpha|\mathbf{V}|\alpha\rangle \\ V|a\rangle &= \mathbf{PV}|\alpha\rangle. \end{aligned} \tag{67}$$

We conclude that it is always possible to interpret the discrete states of a system either as bound states or as elementary-particle states since the formalism presented in this section allows us to switch from the one description to the other.

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Asymptotic Behavior of the Scattering Amplitude and Normal and Abnormal Solutions of the Bethe-Salpeter Equation. II*

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The high-energy asymptotic expansion of the Green's function for the scattering of two scalar particles in the crossed channel (t channel) is investigated in the ladder approximation by using the scalar-photon-exchange model with scalar coupling. It is shown that each term of the asymptotic expansion in the t channel exactly corresponds to the solutions of the Bethe-Salpeter equation for bound states if one considers the expansion in powers of $(-t-m^2+v_0)$ instead of $(-t)$, where m and $v_0^{1/2}$ are the internal mass and the constant external mass ($v_0 \neq m^2$), respectively. It is proved that all normal solutions of the Bethe-Salpeter equation appear in this expansion. The problem of whether or not abnormal solutions also appear in this expansion is analyzed in detail. Exact solutions in some special cases are presented and discussed.

1. INTRODUCTION

IN a previous paper,[1] which we shall refer to as I, we obtained the exact solution to the Bethe-Salpeter-type integral equation for the off-the-mass-shell scattering amplitude in the case $\mu'=0$, $\mu=2m$, and $s=0$, where μ' and μ are the exchanged-meson masses in the kernel and in the inhomogeneous term, respectively, s being the invariant energy; and we investigated its high-energy asymptotic expansion in the crossed channel (t channel). It was found there that in our model the leading term and the second one exactly correspond to the normal solutions of the Bethe-Salpeter equation for bound states with $n=l+1$ and $n=l+2$, respectively, but the third term does not correspond to those with $n=l+3$, where n and l are the principal and the azimuthal quantrum numbers, respectively. Since we artificially introduced a massive meson only in the inhomogeneous term in order to avoid infrared divergence, it was not clear whether or not the above result was owing to the introduction of the special meson.

The purpose of the present paper is to discuss the high-energy asymptotic expansion (in the crossed channel) of the Green's function for the scattering of two scalar particles in the Bethe-Salpeter formalism in the case $\mu'=\mu=0$ and s arbitrary. In spite of its similarity to the equation discussed in I, it is, unfortunately, extremely difficult to find the exact solution (in closed

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[1] N. Nakanishi, Phys. Rev. **135**, B1430 (1964).

form) to the integral equation which we encounter here, except for a very special case which we already discussed elsewhere.[2] However, we can investigate the properties of the asymptotic expansion of the Green's function by making some ansatz. We shall make use of the analogy from the exact solution obtained in I as well as possible.

In the next section, we summarize various forms of the integral equation for the weight function of the perturbation-theoretical integral representation. In Sec. 3, we investigate relations between the coefficients in the asymptotic expansion of the scattering amplitude and the solutions of the Bethe-Salpeter equation. In Sec. 4, the problem of abnormal solutions is discussed in detail. Some conclusions are presented in the final section.

2. PRELIMINARIES

We consider the integral equation

$$(m_1^2-v)(m_2^2-w)f(v,w,t)=\frac{1}{\mu^2-t}+\frac{\lambda}{\pi^2 i}\int d^4p'\frac{f(v',w',t')}{\mu'^2-(p-p')^2},\quad (2.1)$$

with

$$\begin{aligned}(k+q)^2&=v_0, & (k+p)^2&=v, & (k+p')^2&=v',\\ (k-q)^2&=w_0, & (k-p)^2&=w, & (k-p')^2&=w',\quad (2.2)\\ (2k)^2&=s, & (p-q)^2&=t, & (p'-q)^2&=t',\end{aligned}$$

where we have omitted $-i\epsilon$ in the denominators for simplicity. In (2.2), $2k$ stands for the total four-momentum, q, p', and p denoting the relative momenta in the initial, an intermediate, and the final state, respectively. When $s<(m_1+m_2)^2$, the off-the-mass-shell scattering amplitude $f(v,w,t)$ has the perturbation-theoretical integral representation:

$$f(v,w,t)=\int_0^1 dy\int_{-1}^1 dz\int_0^\infty d\gamma \times\frac{\varphi(y,z,\gamma)}{[\gamma+(1-y)\beta(z,v,w)+y(\mu^2-t)]^3},\quad (2.3)$$

with

$$\beta(z,v,w)\equiv\tfrac{1}{2}(1+z)(m_1^2-v)+\tfrac{1}{2}(1-z)(m_2^2-w).\quad (2.4)$$

By a slight generalization of our previous work,[3] we get

$$\varphi(y,z,\gamma)=(1-y)\delta(\gamma)+\lambda\int_0^1 dy'\int_{-1}^1 dz'\int_0^\infty d\gamma' \times K(y,z,\gamma;y',z',\gamma')\varphi(y',z',\gamma'),\quad (2.5)$$

$$\begin{aligned}K(y,z,\gamma;y',z',\gamma')\equiv{}&\tfrac{1}{2}y(1-y)\Theta(y,y',z,z')\int_0^1(1-\xi)^2d\xi\\ &\times\delta'(\xi(1-\xi)y'\gamma-y[(1-\xi)\gamma'+\xi\mu'^2+(1-\xi)^2y'\mu^2\\ &+(1-\xi)^2y'(1-y')\alpha(z')+(1-\xi)^2(1-y')^2\rho(z')]),\end{aligned}\quad (2.6)$$

where

$$\Theta(y,y',z,z')\equiv\theta\left(R(z,z')-\frac{y(1-y')}{y'(1-y)}\right),\quad (2.7)$$

$$R(z,z')\equiv\frac{1\mp z}{1\mp z'}\quad\text{for}\quad z\gtrless z',\quad (2.8)$$

$$\alpha(z)\equiv\tfrac{1}{2}(1+z)(m_1^2-v_0)+\tfrac{1}{2}(1-z)(m_2^2-w_0),\quad (2.9)$$

$$\rho(z)\equiv\tfrac{1}{2}(1+z)m_1^2+\tfrac{1}{2}(1-z)m_2^2-\tfrac{1}{4}(1-z^2)s.\quad (2.10)$$

In particular, in case of $\mu'=0$, the kernel (2.6) reduces to

$$K(y,z,\gamma;y',z',\gamma')=\frac{\tfrac{1}{2}(1-y)\Theta(y,y',z,z')\delta(y'\gamma-y\gamma')}{\gamma'+y'\mu^2+y'(1-y')\alpha(z')+(1-y')^2\rho(z')}.\quad (2.11)$$

Putting

$$\varphi(y,z,\gamma)=\delta(\gamma)(1-y)\phi(y,z),\quad (2.12)$$

we have

$$\phi(y,z)=1+\tfrac{1}{2}\lambda\int_{-1}^1 dz'\int_0^1 dy' \times\frac{y'^{-1}(1-y')\Theta(y,y',z,z')\phi(y',z')}{y'\mu^2+y'(1-y')\alpha(z')+(1-y')^2\rho(z')}.\quad (2.13)$$

If we introduce a variable x through

$$y^{-1}(1-y)=x,\quad (2.14)$$

then $\phi(y,z)\equiv\psi(x,z)$ satisfies

$$\psi(x,z)=1+\tfrac{1}{2}\lambda\int_{-1}^1 dz'\int_0^{xR(z,z')}dx' \times\frac{x'\psi(x',z')}{(1+x')\mu^2+x'\alpha(z')+x'^2\rho(z')}.\quad (2.15)$$

When the denominator of the integrand is positive-definite, (2.15) is a Volterra integral equation so that its Neumann series converges for any value of λ and gives the unique solution. Hence, $\psi(x,z)$ is an entire function of λ and holomorphic in the s plane with a cut $s\geq(m_1+m_2)^2$ if $\mu^2+m_1^2>v_0$ and $\mu^2+m_2^2>w_0$. Furthermore, on account of the positive definiteness of the kernel, we have $\psi(x,z)>1$ for $\lambda>0$.

Now, if we consider the case $m_1=m_2\equiv m$ and $v_0=w_0$, the kernel is invariant under the transformation $z\longrightarrow -z$

[2] N. Nakanishi, Nuovo Cimento (to be published).
[3] N. Nakanishi, Phys. Rev. 133, B214 (1964).

and $z' \to -z'$. This means that $\psi(x,-z)$ is also a solution of (2.15) if $\psi(x,z)$ is so. Thus because of the uniqueness of the solution, $\psi(x,z)$ must be an even function of z. Furthermore, if $s=0$ then the kernel is independent of z and z' except in $R(z,z')$. Defining

$$\psi^{[j]}(x) \equiv \tfrac{1}{2}\int_{-1}^{1} dz\,(1-z^2)^j \psi(x,z)\,, \tag{2.16}$$

and using formulas

$$\begin{aligned}\int_{-1}^{1} dz\,\theta(R(z,z')-X) &= 2(1-X)\theta(1-X)\,,\\ \int_{-1}^{1} dz(1-z^2)\theta(R(z,z')-X) &= [\tfrac{4}{3}(1-X)^2(1+2X)\\ &\quad +2X^2(1-X)(1-z'^2)]\theta(1-X)\,,\end{aligned} \tag{2.17}$$

etc., we obtain a set of reduced equations

$$\psi^{[0]}(x) = 1+\lambda\int_0^x dx'\left(1-\frac{x'}{x}\right)\frac{x'\psi^{[0]}(x')}{\eta(x')}\,, \tag{2.18}$$

$$\begin{aligned}\psi^{[1]}(x) = \tfrac{2}{3}+\lambda\int_0^x dx'\frac{x'}{\eta(x')}\Bigg[&\tfrac{2}{3}\left(1-\frac{x'}{x}\right)^2\left(1+\frac{2x'}{x}\right)\\ &\times\psi^{[0]}(x')+\frac{x'^2}{x^2}\left(1-\frac{x'}{x}\right)\psi^{[1]}(x')\Bigg]\,,\end{aligned} \tag{2.19}$$

etc., where

$$\eta(x) \equiv (1+x)\mu^2 + x(m^2-v_0) + x^2m^2\,. \tag{2.20}$$

The formulas (2.17) can be applied to (2.5) to get a set of reduced equations in case of $\mu'\neq 0$. Of course, if one wants to have the equation for $\psi^{[0]}$ alone, it can easily be obtained directly from (2.1) with $v=w$ and $m_1=m_2$.

3. ASYMPTOTIC EXPANSION

We first consider the case $v_0=w_0=m_1^2=m_2^2$ and $\mu\neq 0$. We denote $\psi(x,z)$ in this case by $\tilde\psi(x,z)$. In particular, when $s=0$ and $\mu=2m$ with $m=1$, (2.15) becomes

$$\tilde\psi(x,z) = 1+\tfrac{1}{2}\lambda\int_{-1}^{1} dz'\int_0^{xR(z,z')} dx'\frac{x'\tilde\psi(x',z')}{(2+x')^2}\,. \tag{3.1}$$

The exact solution to (3.1) was found in I. We shall discuss the case $\mu\neq 2m$ in Appendix A.

Now, for the sake of later convenience, we consider a slightly generalized equation,

$$\tilde\psi(x,z) = 1+\tfrac{1}{2}\lambda\int_{-1}^{1} dz'\int_0^{xR(z,z')} dx'\frac{x'\tilde\psi(x',z')}{(\alpha+x')^2}\,, \tag{3.2}$$

which is obtained by transforming x into $2x/\alpha$. The solution to (3.2) is given by

$$\tilde\psi(x,z) = F\left(-\nu,\nu+1;1;-\frac{x^2(1-z^2)}{4\alpha(\alpha+x)}\right)\,, \tag{3.3}$$

with

$$\nu \equiv (\lambda+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}\,, \tag{3.4}$$

where F stands for a hypergeometric function.

The behavior of the scattering amplitude as $t\to\infty$ is determined by the behavior of the weight function at $y\approx 0$,[3] but it is convenient to consider the asymptotic expansion of the latter in powers of x rather than in powers of y^{-1}. The transformation formula of the hypergeometric function leads to

$$\begin{aligned}\tilde\psi(x,z) &= \frac{\Gamma(2\nu+1)}{[\Gamma(\nu+1)]^2}\left[\frac{x^2(1-z^2)}{4\alpha(\alpha+x)}\right]^\nu\\ &\quad\times F\left(-\nu,-\nu;-2\nu;-\frac{4\alpha(\alpha+x)}{x^2(1-z^2)}\right)+O(x^{-\nu-1})\\ &= \sum_{j=0}^{\infty}(-1)^j\tilde h_j(z)(x/\alpha)^{\nu-j}+O(x^{-\nu-1})\,,\end{aligned} \tag{3.5}$$

with

$$\tilde h_j(z) \equiv \sum_{i=0}^{j}\frac{(-1)^i 2^{-2(\nu-i)}\Gamma(2\nu-i+1)\Gamma(\nu+j-2i)}{i!(j-i)![\Gamma(\nu-i+1)]^2\Gamma(\nu-i)}\times(1-z^2)^{\nu-i}\,. \tag{3.6}$$

We introduce a quantity

$$\tilde f(v,w,t) \equiv \int_0^1 dy\int_{-1}^{1} dz\times\frac{(1-y)\tilde\psi(x,z)}{[(1-y)\beta(z,v,w)+y(2\alpha-t)]^3}\,, \tag{3.7}$$

which, of course, represents a scattering amplitude only when $\alpha=2$. We notice a formula

$$\int_0^1 dy\frac{(1-y)x^a}{[(1-y)A+yB]^3} = \tfrac{1}{2}\Gamma(1-a)\Gamma(a+2)A^{-a-2}B^{a-1}\,, \tag{3.8}$$

where x is defined by (2.14). This formula is valid for $1>\mathrm{Re}\,a>-2$, but if we take Hadamard's finite part[3] in the left-hand side, (3.8) can be used also for $\mathrm{Re}\,a\geq 1$. Applying (3.8) to (3.7) with (3.5), we have

$$\begin{aligned}\tilde f(v,w,t) &= \tfrac{1}{2}\sum_{j=0}^{[\nu+1]}(-1)^j\Gamma(-\nu+j+1)\Gamma(\nu-j+2)\\ &\quad\times\int_{-1}^{1} dz\frac{\tilde h_j(z)}{[\beta(z,v,w)]^{\nu-j+2}}\frac{(2\alpha-t)^{\nu-j-1}}{\alpha^{\nu-j}}+o(t^{-2})\\ &= \sum_{k=0}^{[\nu+1]}\tilde H_k(v,w)(2\alpha)^{-\nu+k}(-t)^{\nu-k-1}+o(t^{-2})\,,\end{aligned} \tag{3.9}$$

with

$$\tilde{H}_k(v,w)\equiv\frac{\pi}{\sin\nu\pi}\cdot\frac{1}{2\Gamma(\nu-k)}\sum_{j=0}^{k}\frac{2^{\nu-j}\Gamma(\nu-j+2)}{(k-j)!}\times\int_{-1}^{1}dz\frac{\tilde{h}_j(z)}{[\beta(z,v,w)]^{\nu-j+2}}. \quad (3.10)$$

We have thus obtained the complete asymptotic expansion of $\tilde{f}(v,w,t)$.

In order to see the nature of $\tilde{h}_j(z)$, we substitute (3.5) and

$$(\alpha+x)^{-2}=x^{-2}\sum_{j=0}^{\infty}(-1)^j(j+1)(\alpha/x)^j \quad (3.11)$$

in (3.2), and compare the coefficients of each power of x. Then it is seen that $\tilde{h}_j(z)$ must be a solution of a homogeneous equation

$$\tilde{h}_j(z)=\frac{\lambda}{2(\nu-j)}\sum_{i=0}^{j}(j-i+1)\times\int_{-1}^{1}dz'[R(z,z')]^{\nu-j}\tilde{h}_i(z). \quad (3.12)$$

Therefore, the quantity

$$\tilde{g}_{\nu l}{}^j(z)\equiv[2^{\nu-j}\Gamma(\nu-j+2)/(k-j)!]\tilde{h}_j(z), \quad (3.13)$$
$$(l\equiv\nu-k-1),$$

appearing in the expression for $\tilde{H}_k(v,w)$ satisfies

$$\tilde{g}_{\nu l}{}^j(z)=\frac{\lambda}{2(\nu-j)}\sum_{i=0}^{j}\frac{j-i+1}{2^{j-i}}\frac{\Gamma(\nu-j+2)(\nu-l-i-1)!}{\Gamma(\nu-i+2)(\nu-l-j-1)!}\times\int_{-1}^{1}dz'[R(z,z')]^{\nu-j}\tilde{g}_{\nu l}{}^i(z'). \quad (3.14)$$

When $\nu=n$ (a positive integer), (3.14) would be identical with the Cutkosky equation[4] if the factor $(j-i+1)/2^{j-i}$ were absent. This factor is equal to unity only when $j=0$ or $j=1$. This is the reason why only the leading term and the second one correspond to the solutions of the Bethe-Salpeter equation.[1] However, from this reasoning alone it cannot be inferred whether or not $\tilde{g}_{nl}{}^j(z)$ $(j=0,1)$ corresponds to normal solutions. From the explicit expression given in I or (3.6), we know that

$$\tilde{g}_{nl}{}^j(z)=\text{const}g_{0nl}{}^j(z) \quad \text{for} \quad j=0,1, \quad (3.15)$$

where $g_{\kappa nl}{}^j(z)$ denotes the Cutkosky function.

The reason why the third term does not correspond to the Cutkosky function seems thus to be due to the fact that the inhomogeneous term in the original equation is different from the kernel, because the unpleasant factor $(j-i+1)$ originates from (3.11). Hence we shall next consider the case $\mu=0$ but $\alpha(z)\neq0$. Then $f(v,w,t)$ is essentially the Green's function apart from the δ-function term and some constant factors. From (2.15) we have

$$\psi(x,z)=1+\frac{\lambda}{2}\int_{-1}^{1}dz'\int_{0}^{xR(z,z')}dx'\frac{\psi(x',z')}{\alpha(z')+x'\rho(z')}. \quad (3.16)$$

It is extremely difficult to solve (3.16) in closed form even if $m_1=m_2\neq v_0=w_0$ and $s=0$. However, we can apply the method mentioned just above to (3.16) in the general case. We introduce an ansatz

$$\psi(x,z)\cong\sum_{j=0}^{[\nu+1]}(-1)^jh_j(z)(x/\alpha)^{\nu-j} \quad (3.17)$$

in analogy with (3.5), but in (3.17) ν is now a function of s. In (3.17), α is an arbitrary constant having the dimension of squared mass, say $\alpha=\alpha(\frac{1}{2})$. Substitution of (3.17) in (3.16) yields

$$h_j(z)=\frac{\lambda}{2(\nu-j)}\sum_{i=0}^{j}\int_{-1}^{1}dz'\frac{[R(z,z')]^{\nu-j}}{[\rho(z')]^{j-i+1}}\times\left[\frac{\alpha(z')}{\alpha}\right]^{j-i}h_i(z'). \quad (3.18)$$

Since

$$f(v,w,t)=\int_{-1}^{1}dz\int_{0}^{1}dy\frac{(1-y)\psi(x,z)}{[(1-y)\beta(z,v,w)+y(-t)]^3}, \quad (3.19)$$

(3.8) leads to

$$f(v,w,t)\cong\tfrac{1}{2}\sum_{j=0}^{[\nu+1]}(-1)^j\Gamma(-\nu+j+1)\Gamma(\nu-j+2)\times\int_{-1}^{1}dz\frac{h_j(z)}{[\beta(z,v,w)]^{\nu-j+2}}\frac{(-t)^{\nu-j-1}}{\alpha^{\nu-j}}. \quad (3.20)$$

As is easily seen, the coefficients of $(-t)^{\nu-j-1}$ do *not* directly relate to the solutions of the Bethe-Salpeter equation except for the leading term $j=0$. Thus the simple analogy from the Regge-pole theory in the nonrelativistic potential scattering seems not to be valid in field theory.

In order to find some relation with the solution of the Bethe-Salpeter equation, we consider the special case $m_1=m_2\neq v_0=w_0$ hereafter. Then

$$\alpha(z)=m^2-v_0=\alpha. \quad (3.21)$$

An analogous function to $\tilde{H}_k(v,w)$ given in (3.10) can be obtained if we consider an expansion in powers of $(-t-\alpha)$ instead of $(-t)$:

$$f(v,w,t)\cong\sum_{k=0}^{[\nu+1]}H_k(v,w)\alpha^{-\nu+k}(-t-\alpha)^{\nu-k-1}, \quad (3.22)$$

with

$$H_k(v,w)\equiv\frac{\pi}{\sin\nu\pi}\cdot\frac{1}{2\Gamma(\nu-k)}\sum_{j=0}^{k}\frac{\Gamma(\nu-j+2)}{(k-j)!}\times\int_{-1}^{1}dz\frac{h_j(z)}{[\beta(z,v,w)]^{\nu-j+2}}. \quad (3.23)$$

[4] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

Then the function

$$g_{\nu l}{}^{j}(z)\equiv[\Gamma(\nu-j+2)/(k-j)!]h_j(z) \qquad (l\equiv\nu-k+1) \quad (3.24)$$

appearing in (3.23) satisfies

$$g_{\nu l}{}^{j}(z)=\frac{\lambda}{2(\nu-j)}\sum_{i=0}^{j}\frac{\Gamma(\nu-j+2)(\nu-l-i-1)!}{\Gamma(\nu-i+2)(\nu-l-j-1)!}\times\int_{-1}^{1}dz\frac{[R(z,z')]^{\nu-j}}{[\rho(z')]^{j-i+1}}g_{\nu l}{}^{i}(z'). \quad (3.25)$$

When $\nu=n$, (3.25) is exactly identical with the Cutkosky equation. Thus $H_k(v,w)$ exactly corresponds to the solutions of the Bethe-Salpeter equation with $n=l+k+1$. But from this reasoning alone we cannot infer whether these solutions are normal or abnormal. To determine this, it is sufficient to consider a special case $s=0$. In this case, since (3.16) and (3.2) have precisely the same asymptotic form for large x in their iterative expansions, the leading term of $\psi(x,z)$ should be equal to that of $\tilde{\psi}(x,z)$ in the weak coupling limit $\lambda\to 0$.[5] Therefore, the functions $H_k(v,w)$ must be *normal* solutions.

The above conclusion does not, however, exclude the possible appearance of abnormal solutions in the asymptotic expansion of $\psi(x,z)$, because instead of (3.17) we should take a more general ansatz

$$\psi(x,z)\cong\sum_{\kappa=0}^{\infty}\sum_{j=0}^{[\nu_\kappa+1]}(-1)^j h_{\kappa j}(z)(x/\alpha)^{\nu_\kappa-j}, \quad (3.26)$$

where $h_{\kappa j}(z)$ satisfies (3.18). We cannot easily see whether or not the series of abnormal solutions ($\kappa>0$) are present. But since $\psi(x,z)$ is an even function of z, the abnormal solutions of odd κ cannot appear in (3.26).

4. PROBLEM OF ABNORMAL SOLUTIONS

In order to investigate whether or not the series of $\kappa=2$ exists in (3.26), we shall consider the reduced equations (2.18) and (2.19) in this section. Considering the case $m_1=m_2=1$, $v_0=w_0$, and $s=0$, we have

$$\psi^{[0]}(x)=1+\lambda\int_0^x dx'\left(1-\frac{x'}{x}\right)\frac{\psi^{[0]}(x')}{\alpha+x'}. \quad (4.1)$$

The solution to this equation was already given in our short note[2]:

$$\psi^{[0]}(x)=F(-\nu,\nu+1;2;-x/\alpha), \quad (4.2)$$

where ν is given by (3.4), and $\alpha=1-v_0$. The asymptotic expansion of $\psi^{[0]}(x)$ is

$$\psi^{[0]}(x)=\sum_{j=0}^{[\nu+1]}\frac{\Gamma(2\nu-j+1)}{\Gamma(\nu-j+1)\Gamma(\nu-j+2)j!}\times\left(\frac{x}{\alpha}\right)^{\nu-j}+o(x^{-1}). \quad (4.3)$$

Considering the forward scattering in the t channel in which $v=w$, we have

$$f(v,v,t)=\sum_{j=0}^{[\nu+1]}\frac{\Gamma(2\nu-j+1)\Gamma(-\nu+j+1)}{\Gamma(\nu-j+1)j!}\times\frac{(-t)^{\nu-j-1}}{\alpha^{\nu-j}[\beta(v)]^{\nu-j+2}}+o(t^{-2})$$
$$=\sum_{k=0}^{[\nu+1]}H_k(v)\alpha^{-\nu+k}(-t-\alpha)^{\nu-k-1}+o(t^{-2}), \quad (4.4)$$

where

$$H_k(v)\equiv\frac{\pi}{\sin\nu\pi}\frac{1}{\Gamma(\nu-k)}\times\sum_{j=0}^{k}\frac{(-1)^j\Gamma(2\nu-j+1)}{j!(k-j)!\Gamma(\nu-j+1)}\frac{1}{[\beta(v)]^{\nu-j+2}}, \quad (4.5)$$

with $\beta(v)\equiv 1-v$. It is easy to see that the functions

$$\mathcal{Y}_{lm}(\mathbf{p})\sum_{j=0}^{n-l-1}\frac{g_{0nl}{}^{j}}{[\beta(v)]^{n-j+2}} \quad (4.6)$$

are the normal solutions of the corresponding Bethe-Salpeter equation, where $\mathcal{Y}_{lm}(\mathbf{p})$ stands for a solid harmonic, and

$$g_{0nl}{}^{j}\equiv(-1)^j(2n-j)!/j!(n-j)!(n-l-j-1)!. \quad (4.7)$$

Thus the normal solutions only appear in the asymptotic expansion of $\psi^{[0]}(x)$. From this result, however, we *cannot* conclude that the same is true for $\psi(x,z)$, because the abnormal solutions vanish when integrated over z from -1 to $+1$ as is seen from

$$\int_{-1}^{1}dz(1-z^2)^\nu C_\kappa{}^{\nu+1/2}(z)=0 \quad \text{for} \quad \kappa=1,2,\cdots, \quad (4.8)$$

where $C_\kappa{}^{\mu}(z)$ denotes a Gegenbauer polynomial. Since abnormal solutions have explicit dependence on p_0 when the total four-momentum vanishes, they cannot appear in the asymptotic expansion of the forward scattering amplitude even in the case $\mu'\neq 0$ because of Lorentz invariance.

In order to explore the problem of abnormal solutions,

[5] This reasoning is not rigorous. We shall explicitly verify this statement by special cases in the next section.

therefore, we must consider $\psi^{[1]}(x)$ at least:

$$\psi^{[1]}(x)=\tfrac{2}{3}+\lambda\int_0^x \frac{dx'}{\alpha+x'}\left[\frac{2}{3}\left(1-3\frac{x'^2}{x^2}+2\frac{x'^3}{x^3}\right)\psi^{[0]}(x')+\left(\frac{x'^2}{x^2}-\frac{x'^3}{x^3}\right)\psi^{[1]}(x')\right], \quad (4.9)$$

where $\psi^{[0]}(x)$ is given by (4.2). As shown in Appendix B, the exact solution to (4.9) is given by

$$\psi^{[1]}(x)=\varphi(x)+x^{-2}\psi^{[0]}(x)\chi(x), \quad (4.10)$$

with

$$\varphi(x)\equiv\tfrac{2}{3}[F(-\nu,\nu+1;4;-x/\alpha)-(x/\alpha)F'(-\nu,\nu+1;4;-x/\alpha)], \quad (4.11)$$

and

$$\chi(x)\equiv\int_0^x dx'x'^{-2}[\psi^{[0]}(x')]^{-2}\times\int_0^{x'}dx''\frac{\lambda x''^3}{\alpha+x''}\psi^{[0]}(x'')\varphi(x''). \quad (4.12)$$

After somewhat tedious calculation, we obtain the asymptotic expansion of $\psi^{[1]}(x)$:

$$\psi^{[1]}(x)=\frac{2\Gamma(2\nu+1)}{(2\nu+3)[\Gamma(\nu+1)]^2}\left(\frac{x}{\alpha}\right)^\nu\left[1+\frac{\nu(\nu+2)}{2(\nu+1)}\left(\frac{\alpha}{x}\right)+\frac{\nu}{2(2\nu-1)(2\nu+1)}\left(\frac{\alpha}{x}\right)^2\ln\left(\frac{x}{\alpha}\right)+O(x^{-2})\right]. \quad (4.13)$$

For the sake of comparison, we write here the asymptotic expansions of

$$\tilde{\psi}^{[j]}(x)\equiv\tfrac{1}{2}\int_{-1}^1 dz(1-z^2)^j\tilde{\psi}(x,z) \quad (4.14)$$

for $j=0, 1$.

$$\tilde{\psi}^{[0]}(x)=\frac{1}{2\nu+1}\left(\frac{x}{\alpha}\right)^\nu\left[1+\frac{\nu+1}{2}\left(\frac{2\alpha}{x}\right)+\frac{\nu(\nu+1)}{8}\left(\frac{2\alpha}{x}\right)^2+O(x^{-3})\right],$$

$$\tilde{\psi}^{[1]}(x)=\frac{2(\nu+1)}{(2\nu+1)(2\nu+3)}\left(\frac{x}{\alpha}\right)^\nu\times\left[1+\frac{\nu(\nu+2)}{2(\nu+1)}\left(\frac{2\alpha}{x}\right)+O(x^{-2})\right]. \quad (4.15)$$

Comparing (4.3) and (4.13) with (4.15), we see

$$\psi_0^{[0]}(x)/\tilde{\psi}_0^{[0]}(x)=\psi_0^{[1]}(x)/\tilde{\psi}_0^{[1]}(x)=\Gamma(2\nu+2)/\Gamma(\nu+1)\Gamma(\nu+2), \quad (4.16)$$

where the subscript 0 indicates to take the leading term. The value in the right-hand side of (4.16) tends to unity in the weak coupling limit $\lambda\to 0$ as is expected. As for the second terms, their ratios to the respective leading terms coincide for $\psi^{[j]}(x)$ and $\tilde{\psi}^{[j]}(x)$ if we make α in the former correspond to 2α in the latter [cf. (3.7)]. Thus we may infer that the leading and the second term in the asymptotic expansion of $f(v,w,t)$ in powers of $(-t-\alpha)$ correspond to the normal solutions with $n=l+1$ and those with $n=l+2$, respectively.

Now the problem is the third term. The appearance of a logarithmic function in the third term of (4.13) is remarkable. It gives a term behaving like $(-t)^{\nu-3}\ln(-t)$ in the asymptotic expansion of $f(v,w,t)$.[6] It corresponds to a double Regge pole in the complex angular momentum plane. The appearance of a logarithmic function contradicts our ansatz (3.17) or (3.26). Such a situation can happen only when the Cutkosky equation (3.25) has no solution. Indeed, we shall show in Appendix C that $g_{0nl}{}^2(z)$ is divergent at $s=0$. This peculiar situation is, of course, due to the degeneracy with the abnormal solution with $\kappa=2$.

We may thus infer that the asymptotic expansion of $\tilde{\psi}(x,z)$ is much disturbed by the presence of abnormal solutions. Unfortunately, it is very difficult to show explicitly the existence of the series of the abnormal solutions with $\kappa=2$ in (3.26). Instead, we shall content ourselves by showing that $\tilde{\psi}(x,z)$ indeed has the contribution from the abnormal solution with $\kappa=2$.

In order to avoid the difficulty of degeneracy at $s=0$, we consider the solution up to the first order of s. Since $\tilde{g}_{\nu l}{}^0(z)$ and $\tilde{g}_{\nu l}{}^1(z)$ satisfy the Cutkosky equation (3.25), we easily obtain

$$\lambda=\nu(\nu+1)-\frac{\nu(\nu+1)^2}{2(2\nu+3)}s+O(s^2),$$

$$\tilde{g}_{\nu l}{}^0(z)=c\left[(1-z^2)^\nu+\frac{\nu(\nu+1)}{8(2\nu+3)}(1-z^2)^{\nu+1}s\right]+O(s^2),$$

$$\tilde{g}_{\nu l}{}^1(z)=c\left\{\frac{\nu(\nu-l-1)}{2(\nu+1)}[(1-z^2)^\nu-2(1-z^2)^{\nu-1}]+\frac{\nu(\nu-l-1)}{16(\nu+1)(2\nu+3)}[(\nu+1)(\nu+2)(1-z^2)^{\nu+1}-2(\nu^2-2)(1-z^2)^\nu-4(2\nu+3)(1-z^2)^{\nu-1}]s\right\}+O(s^2), \quad (4.17)$$

where c is a certain s-dependent constant. In the above calculation, we have used a formula

$$\int_{-1}^1 dz'[R(z,z')]^\mu(1-z'^2)^{\mu+k}=\frac{\mu\, k!\Gamma(\mu+k+1)}{\Gamma(\mu+2k+2)}\times\sum_{j=0}^k\frac{2^{2k-2j+1}\Gamma(\mu+2j)}{j!\Gamma(\mu+j+1)}(1-z^2)^{\mu+j}, \quad (4.18)$$

[6] This can be seen most easily by differentiating (3.8) with respect to a.

where k is a nonnegative integer, μ being arbitrary. The proof of (4.18) is given in Appendix D.

Now, as is easily derived from (2.15), $\tilde{g}_{\nu l}^2(z)$ for general s satisfies

$$\tilde{g}_{\nu l}^2(z)=\frac{\lambda}{2(\nu-2)}\sum_{i=0}^{2}\frac{\Gamma(\nu)(\nu-l-i-1)!}{\Gamma(\nu-i+2)(\nu-l-3)!}\times\int_{-1}^{1}dz'[R(z,z')]^{\nu-i}f_i(z')\tilde{g}_{\nu l}^i(z'),\quad(4.19)$$

where

$$\begin{aligned}f_0(z)&\equiv-\tfrac{1}{4}[\rho(z)]^{-2}+[\rho(z)]^{-3},\\ f_1(z)&\equiv[\rho(z)]^{-2},\\ f_2(z)&\equiv[\rho(z)]^{-1}.\end{aligned}\quad(4.20)$$

Making an ansatz

$$\tilde{g}_{\nu l}^2(z)=c(\nu-l-1)(\nu-l-2)\left\{\tfrac{1}{8}(1-z^2)^\nu+b\left[(1-z^2)^{\nu-1}-\frac{2(\nu-1)}{2\nu-1}(1-z^2)^{\nu-2}\right]+\sum_{j=0}^{3}a_j(1-z^2)^{\nu-j+1}s\right\}+O(s^2),\quad(4.21)$$

and using (4.18), we obtain

$$\begin{aligned}a_0&=(\nu+2)(\nu+3)/2^6(2\nu+3),\\ a_1&=[-(\nu-2)+2\nu(\nu+1)b]/16(2\nu+3),\\ b&=-(2\nu-1)(2\nu-3)/8(\nu-1)(\nu+1);\end{aligned}\quad(4.22)$$

a_2 is indefinite if we do not consider the second order of s, a_3 depending on a_2. One should notice that the quantity in the square bracket of (4.21) is just proportional to the abnormal solution with $\kappa=2$ at $s=0$. On the other hand, (3.13) together with (3.6) gives

$$\tilde{g}_{\nu l}^2(z)=\text{const}\left\{\tfrac{1}{8}(1-z^2)^\nu-\frac{\nu-1}{2(\nu+1)}\times\left[(1-z^2)^{\nu-1}-\frac{2(\nu-1)}{2\nu-1}(1-z^2)^{\nu-2}\right]\right\}.\quad(4.23)$$

Thus (4.21) at $s=0$ is not equal to (4.23). The difference between them is nothing but the contribution from the abnormal solution with $\kappa=2$.

5. CONCLUSIONS

In this paper, we have investigated the relation between the asymptotic expansion of the Green's function in the crossed channel and the solutions of the Bethe-Salpeter equation by using the scalar-photon-exchange model. Our conclusions are as follows.

(1) In the case $m_1=m_2\neq v_0=w_0$, we have the exact correspondence between the asymptotic expansion of the Green's function and the solutions of the Bethe-Salpeter equation *if we consider the expansion in powers of* $(-t-\alpha)$ *instead of* $(-t)$, where $\alpha=m^2-v_0$.

(2) All normal solutions appear in this expansion.

(3) No abnormal solutions with odd κ appear.

(4) Abnormal solutions with even κ seem to appear in the asymptotic expansion, but they disappear if one considers the forward scattering only.

(5) When the Cutkosky equation has no solution on account of degeneracy, a logarithmic function of t can appear in the asymptotic expansion.

(6) In the case $m_1\neq m_2$ or $v_0\neq w_0$, it seems to be difficult to establish some relation between the asymptotic expansion and the solutions of the Bethe-Salpeter equation except for the leading term.[3]

Finally, we note that as a by-product of the expression for λ given in (4.17) we explicitly obtain the gradient at $s=0$ of the Regge trajectories corresponding to normal solutions. We find

$$\left.\frac{d\nu}{ds}\right|_{s=0}=\frac{\nu(\nu+1)^2}{2(2\nu+1)(2\nu+3)},\quad(5.1)$$

where ν is given by (3.4). Since

$$\nu=l+k+1\ (k=0,1,2,\cdots),$$

all these Regge trajectories have the same positive slope (5.1).

APPENDIX A: SOLUTION IN THE CASE $\mu\neq 2m$

It is not easy to find the solution in closed form in the case $\mu\neq 2m$ even if $s=0$. Hence, we shall consider a reduced equation only:

$$\tilde{\psi}^{[0]}(x)=1+\lambda\int_0^x dx'\left(1-\frac{x'}{x}\right)\frac{x'\tilde{\psi}^{[0]}(x')}{(1+x')\mu^2+x'^2},\quad(\mathrm{A}1)$$

where we put $m=1$. We can easily transform (A1) into a differential equation

$$\frac{d^2\tilde{\psi}^{[0]}}{dx^2}+\frac{2}{x}\frac{d\tilde{\psi}^{[0]}}{dx}-\frac{\lambda}{(1+x)\mu^2+x^2}\tilde{\psi}^{[0]}=0,\quad(\mathrm{A}2)$$

with

$$\tilde{\psi}^{[0]}(0)=1,\quad(\mathrm{A}3)$$

$$\tilde{\psi}^{[0]\prime}(0)=0.\quad(\mathrm{A}4)$$

It turns out that (A4) is actually unnecessary to determine $\tilde{\psi}^{[0]}(x)$. As was already seen in I, the solution to (A2) with (A3) in case of $\mu=2$ is

$$\tilde{\psi}_{\mu=2}^{[0]}(x)=F(-\nu,\nu+1;\tfrac{3}{2};-x^2/8(2+x))\quad(\mathrm{A}5)$$

with

$$\nu\equiv(\lambda+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}.\quad(\mathrm{A}6)$$

In the case $\mu\neq 2$, by considering $x\tilde{\psi}^{[0]}$, (A2) can be transformed into a hypergeometric equation. After some arrangement, for $0<\mu<2$, we have

$$\tilde{\psi}^{[0]}(x)=\frac{[(1+x)\mu^2+x^2]^{1/2}\,\mathrm{Im}[-P_\nu^1(-\xi_0)P_\nu^1(\xi)]}{x\,\mathrm{Re}[P_\nu^1(-\xi_0)P_\nu^2(\xi_0)]}, \quad \text{(A7)}$$

with

$$\xi\equiv i(\mu^2+2x)/(4\mu^2-\mu^4)^{1/2},$$
$$\xi_0\equiv i\mu^2/(4\mu^2-\mu^4)^{1/2}, \quad \text{(A8)}$$

where $i=(-1)^{1/2}$ and P_ν^j denotes an associated Legendre function of the first kind. It is easy to verify that (A7) satisfies (A4). For the case $\mu>2$, we have only to continue (A7) analytically with respect to μ.

If we consider the limit $\mu\to 0$, $\tilde{\psi}^{[0]}(x)$ is of course divergent. The main term is

$$\tilde{\psi}^{[0]}(x)=\frac{2^\nu\Gamma(\frac{1}{2}\nu+1)\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+2)\Gamma(\frac{1}{2}\nu+\frac{1}{2})}\left(\frac{x}{\mu}\right)^\nu+O(\mu^{-\nu+1}). \quad \text{(A9)}$$

Using (3.8), we have

$$\tilde{f}(v,v,t)=\frac{2^\nu\Gamma(\frac{1}{2}\nu+1)\Gamma(\nu+\frac{1}{2})\Gamma(-\nu+1)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2})\mu^\nu}\times\frac{(-t)^{\nu-1}}{(1-v)^{\nu+2}}+O(\mu^{-\nu+1}). \quad \text{(A10)}$$

If (A1) is solved by an iteration method, and if we take the leading part of each term as $\mu\to 0$, then we have

$$\tilde{\psi}^{[0]}(x)\sim\sum_{n=0}^{\infty}(\lambda^n/n!)[\ln(x/\mu)]^n=(x/\mu)^\lambda, \quad (x\neq 0) \quad \text{(A11)}$$

a result which is correct in the weak coupling limit $\lambda\to 0$.

In the above, we have considered the scattering amplitude by first putting $v_0=1$ and next taking the limit $\mu\to 0$. If we first put $\mu=0$ and later take the limit $v_0\to 1$, from (4.3) we obtain

$$\psi^{[0]}(x)=\frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)\Gamma(\nu+2)}\times\left[\frac{x}{1-v_0}\right]^\nu+O((1-v_0)^{-\nu+1}). \quad \text{(A12)}$$

APPENDIX B: DERIVATION OF (4.10)

We first calculate

$$H(x)\equiv\lambda\int_0^x\frac{dx'}{\alpha+x'}\left(1-3\frac{x'^2}{x^2}+2\frac{x'^3}{x^3}\right)\psi^{[0]}(x'). \quad \text{(B1)}$$

Inserting the expansion of $(\alpha+x)^{-1}$ and

$$\psi^{[0]}(x)=F(-\nu,\nu+1;2;-x/\alpha)$$
$$=\sum_{k=0}^{\infty}\frac{\prod_{j=0}^{k-1}[\lambda-j(j+1)]}{k!(k+1)!}\left(\frac{x}{\alpha}\right)^k \quad \text{(B2)}$$

into (B1), and using a formula given in (A22) of I, we obtain

$$H(x)=6\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{n-1}[\lambda-j(j+1)]}{(n+2)(n+3)(n!)^2}\left(\frac{x}{\alpha}\right)^n, \quad \text{(B3)}$$

whence

$$\int_0^x H(x')dx'=x[F(-\nu,\nu+1;4;-x/\alpha)-1]. \quad \text{(B4)}$$

Therefore,

$$\tfrac{2}{3}[1+H(x)]=\varphi(x), \quad \text{(B5)}$$

where $\varphi(x)$ is given in (4.11). Hence (4.9) is rewritten as

$$\psi^{[1]}(x)=\varphi(x)+\lambda\int_0^x\frac{dx'}{\alpha+x'}\left(\frac{x'^2}{x^2}-\frac{x'^3}{x^3}\right)\psi^{[1]}(x'), \quad \text{(B6)}$$

which is easily transformed into a differential equation

$$\left[\frac{d^2}{dx^2}+\frac{6}{x}\frac{d}{dx}+\frac{6}{x^2}-\frac{\lambda}{x(\alpha+x)}\right]\times[\psi^{[1]}(x)-\varphi(x)]=\frac{\lambda\varphi(x)}{x(\alpha+x)}, \quad \text{(B7)}$$

with

$$\psi^{[1]}(0)=\varphi(0)=\tfrac{2}{3}. \quad \text{(B8)}$$

Putting

$$\hat{\psi}(x)\equiv x^2[\psi^{[1]}(x)-\varphi(x)], \quad \text{(B9)}$$

we have

$$x(\alpha+x)\hat{\psi}''(x)+2(\alpha+x)\hat{\psi}'(x)-\lambda\hat{\psi}(x)=\lambda x^2\varphi(x). \quad \text{(B10)}$$

The homogeneous part of (B10) is satisfied by $\psi^{[0]}(x)$. Hence the function

$$\chi(x)\equiv\hat{\psi}(x)/\psi^{[0]}(x) \quad \text{(B11)}$$

satisfies

$$x(\alpha+x)\psi^{[0]}(x)\chi''(x)+2(\alpha+x)[\psi^{[0]}(x)+x\psi^{[0]\prime}(x)]\chi'(x)=\lambda x^2\varphi(x). \quad \text{(B12)}$$

Since (B12) is a linear differential equation of the first order for $\chi'(x)$, it can be solved by the standard method.

APPENDIX C: DIVERGENCE OF THE CUTKOSKY SOLUTION

We consider

$$h_j^{(k)}\equiv\frac{(n-l-j-1)!}{(n-j+1)!}\int_{-1}^{1}dz\,z^k g_{0nl}{}^j(z) \quad \text{(C1)}$$

in case of $s=0$. Then the Cutkosky equation yields

$$h_j^{(0)}=\frac{n(n+1)}{(n-j)(n-j+1)}\sum_{i=0}^{j}h_i^{(0)}, \quad \text{(C2)}$$

$$h_j^{(2)}=\frac{n(n+1)}{(n-j+2)(n-j+3)}\times\sum_{i=0}^{j}\left[\frac{2h_i^{(0)}}{(n-j)(n-j+1)}+h_i^{(2)}\right]. \quad \text{(C3)}$$

The solution to (C2) is[2]

$$h_j^{(0)}=(-1)^j(2n-j)!/j!(n-j)!(n-j+1)! \quad \text{(C4)}$$

apart from an arbitrary common factor. Substitution of (C2) and (C4) in (C3) yields

$$h_0^{(2)}=(2n)!/(2n+3)n!(n+1)!, \quad \text{(C5)}$$

$$h_1^{(2)}=-(n+3)(2n-1)!/(2n+3)(n-1)!(n+1)!, \quad \text{(C6)}$$

and

$$h_2^{(2)}=-[2(2n-2)!/(2n+3)(n-1)!(n+1)!]+h_2^{(2)}. \quad \text{(C7)}$$

The last equation leads us to $h_2^{(2)}=\infty$. Thus $g_{0nl}^{2}(z)$ cannot be an integrable function.

APPENDIX D: PROOF OF (4.18)

$$\begin{aligned}J_{\mu k}(z)&\equiv\int_{-1}^{1}dz'[R(z,z')]^{\mu}(1-z'^2)^{\mu+k}\\&=(1-z)^{\mu}\int_{-1}^{z}dz'(1+z')^{\mu+k}(1-z')^{k}+(z\to -z)\\&=(1-z^2)^{\mu}\sum_{j=0}^{k}(-1)^j2^{k-j}\\&\qquad\times{}_kC_j(\mu+k+j+1)^{-1}K_{k+j+1}(z),\end{aligned} \quad \text{(D1)}$$

where

$$K_m(z)\equiv(1+z)^m+(1-z)^m. \quad \text{(D2)}$$

We shall first show that (D2) can be rewritten as

$$K_m(z)=\sum_{j=0}^{[m/2]}(-1)^j\frac{m\cdot 2^{m-2j}(m-j-1)!}{j!(m-2j)!}(1-z^2)^j, \quad \text{(D3)}$$

where $m(m-j-1)!=0!=1$ for $m=j=0$.

The right-hand side of (D3), which is denoted by $\hat{K}_m(z)$, can be written in the form

$$\hat{K}_m(z)=\sum_{k=0}^{m}K_k\cdot(1+z)^k(1-z)^{m-k} \quad \text{(D4)}$$

by multiplying it by $1=\{\frac{1}{2}[(1+z)+(1-z)]\}^{m-2j}$. Here,

$$K_k\equiv\sum_{j=0}^{k'}(-1)^j\frac{m(m-j-1)!}{j!(k-j)!(m-k-j)!}, \quad \text{(D5)}$$

with $k'\equiv\min(k, m-k)$. Since $K_{m-k}=K_k$, it is sufficient to consider K_k for $k\leqq[m/2]$. Comparing (D5) with the expansion formula of a hypergeometric function, we find

$$\begin{aligned}K_k&=[m!/k!(m-k)!]F(-k,-m+k;-m+1;1)\\&=[m!/k!(m-k)!][\Gamma(1-k)\prod_{j=1}^{k}(-m+j)]^{-1}\\&=\delta_{0k}.\end{aligned} \quad \text{(D6)}$$

Substitution of (D6) in (D4) leads to $\hat{K}_m(z)=K_m(z)$.

Now, (D1) together with (D3) yields

$$J_{\mu k}(z)=\sum_{j=0}^{k}\frac{2^{2k-2j+1}}{j!}L_j(1-z^2)^{\mu+j}, \quad \text{(D7)}$$

with

$$L_j\equiv\sum_{i=0}^{k}\frac{(-1)^{j+i}{}_kC_i(k+i+1)(k-j+i)!}{(\mu+k+i+1)\,(k-2j+i+1)!}. \quad \text{(D8)}$$

Here, if $2j>k+1$, the terms of $i<2j-k-1$ give no contribution because their denominators are infinite.

$$L_j=L_{j0}+jL_{j1}, \quad \text{(D9)}$$

where[7]

$$\begin{aligned}L_{jm}&\equiv\sum_{i=0}^{k}\frac{(-1)^{j+i}{}_kC_i(k-j+i+1-m)!}{(\mu+k+i+1)(k-2j+i+1)!}\\&=(-1)^j\sum_{i=0}^{k}(-1)^i{}_kC_i\\&\qquad\times\int_0^1dx\cdot x^{\mu+2j-1}\left(\frac{d}{dx}\right)^{j-m}x^{k-j+i+1-m}\\&=\frac{\Gamma(\mu+2j)}{\Gamma(\mu+j+m)}\int_0^1dx\,x^{\mu+k}(1-x)^k\\&=\frac{k!\Gamma(\mu+k+1)\Gamma(\mu+2j)}{\Gamma(\mu+2k+2)\Gamma(\mu+j+m)}.\end{aligned} \quad \text{(D10)}$$

Hence,

$$L_j=\frac{\mu\cdot k!\Gamma(\mu+k+1)\Gamma(\mu+2j)}{\Gamma(\mu+2k+2)\Gamma(\mu+j+1)}. \quad \text{(D11)}$$

Thus (D7) together with (D11) gives (4.18).

---

[7] We may assume $j\geq m$ because when $j=0$ we need L_{j0} only.